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About real interpolation

The origin of the theory of interpolation can be traced back to Marcinkiewicz and the Riesz-Thorin theorem, which states that if a linear function is continuous on L^p and L^q, then it is also continuous on L^r for r between p and q. Later, as it was shown that Sobolev spaces were constituted of functions that have a non-integer order of differentiability, various techniques were conceived to generate similar spaces. Among them were the interpolation methods which have been generalized using a function parameter. Most of the times, we start from the K-method. Let A_0 and A_1 be two Banach spaces continuously embedded into a Hausdorff topological vector space so that $A_0 \cap A_1$ and $A_0 + A_1$ are well defined Banach spaces. One defines the K-functional by

$$K(t,a) := \inf_{a=a_0+a_1} \{\|a_1\|_{A_0} + t\|a_1\|_{A_1}\},$$

for $t > 0$ and $a \in A_0 + A_1$. Given $0 < \theta < 1$ and $q \in [1,\infty]$, a belongs to the interpolation space $(A_0, A_1)_{\theta,q}$ if $a \in A_0 + A_1$ and

$$(2^{-j\theta} K(2^j, a))_{j\in\mathbb{Z}} \in \ell^q.$$

The generalized version is obtained by replacing the sequence $(2^{-j\theta})_{j\in\mathbb{Z}}$ appearing in the expression above with a Boyd function. The J-method is defined in a similar way: one sets

$$J(t,a) := \max\{\|a\|_{A_0}, t\|a\|_{A_1}\},$$

for $t > 0$ and $a \in A_0 \cap A_1$. This time, one considers

$$(2^{-j\theta} J(2^j, b_j))_{j\in\mathbb{Z}} \in \ell^q,$$

with $a = \sum_{j\in\mathbb{Z}} b_j$ and $b_j \in A_0 \cap A_1$ (for all j), the convergence being in $A_0 + A_1$. This approach can be generalized in the same way and one can show that both methods give rise to the same spaces.

In this work, we show that the Boyd functions form a natural apparatus for studying function spaces and interpolation methods with a function parameter provide an interesting tool in this context. For example, they lead to a definition of the Besov spaces of generalized smoothness based on the usual Sobolev spaces.